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On the Separation of Gravitation and Inertia in the Case of Free Motion

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tion of north- and south-going passes) displayed in Fig. 3. Two of them showed too many local reflections to be of any further use, therefore three new sites were defined along the particular subtracks. As an example the radar altimeter returns for an ERS-1 overpass at RAMS (Ramsau / Dachstein) are shown in Fig. 4. All sites were connected to the International Terrestrial Reference Frame by preceding or on-site GPS-measurements, the latter also being used for the estimate of the ionospheric corrections. Special attention is paid to the cross-over point LASS, situated in the near vicinity of the Graz laser station (7 km) for which altimeter derived heights and laser distances are highly correlated and can be used for a direct calibration of the ERS-2 altimeter.

6. Current Status and Future Plans

The recent measurements have shown a substantial decrease of the power of the emitted return pulses by a factor of 8 compared to the first measurements in Schindlet, which complicates the data reduction for noisy sites. The reason may be the failure of one amplifier inside the transponder which is presently investigated. After some test-measurements near the observatory Lustbühel it is planned to repeat the measurements in Austria for a further 70 days period. After that the Graz transponder will be employed, together with the Copenhagen transponder, for a dedicated mission which aims at the connection of North Sea and the Adriatic on the one hand and the connection of the individual sea surfaces to the coastland on the other.

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Abstract

The authors explored the possibility of separating gravitation from inertia in the case of free motion according to general relativity, proposed a general method of determining the relativistic gravity field of the earth, and put forward and proved two important statements.

Zusammenfassung

Die Verfasser untersuchten die Möglichkeit der Trennung von Gravitation und Trägheit in dem Fall der freien Bewegung gemäß der allgemeinen Relativitätstheorie. Es wurde eine allgemeine Methode zur Berechnung des relativistischen Gravitätsfeldes der Erde vorgeschlagen. Weiters wurden zwei wichtige Theoreme aufgestellt und bewiesen.

1. Introduction

Quite a few geodesists paid attention to relativistic effects in geodesy [2, 5, 7, 8, 12]. It is generally agreed that, if an order 10^{-8}–10^{-9} or a higher accuracy requirement is needed, the relativistic effects should be considered.

When a particle is in the state of motion in a gravitational field, it experiences some forces, which generally include gravitational andinertial forces. Some forces may be balanced by each other, and others may not. In the special case that the particle is moving freely in the gravitational field, the nature of the particle is very important. In this case, the particle will not sense any force or the forces it senses are completely balanced so that it senses a resultant zero, because of *Einstein's equivalence principle,* which states that *the gravitational mass is equivalent to the inertial mass,* and later generalized as follows: *in a closed freely falling system, one cannot find out whether the system is in the state of free fall in a gravitational field or in the state of uniform motion or at rest far away from any matter sources* [16]. Hence, generally it is believed that gravitation and inertia are not distinguishable. But this belief is correct only if one considers the force at one point only. In a finite region however, gravitation and inertia can be separated, at least in principle, because the gravitational field is essentially different from the "inertial field". Roughly speaking, the inertial field is smoother and more regular than the gravitational field, so that we can find some kind of quantity which is sensitive only to the gravitational effects. This quantity is the Riemann tensor, which has an absolute character. We can conclude that there is a gravitational field or none, according as the Riemann tensor does not vanish or vanishes. In the case of free motion, if we can find a way to determine the Riemann tensor, then we have separated gravitation from inertia, and in this sense, gravitation and inertia are absolutely distinguishable. This conclusion was first pointed out by Synge [14], and later followed a detailed study emphasizing the application in geodesy by Moritz [7]. However, unfortunately, we have not yet reached a final confirmation. The key problem is: is it possible to find the Riemann tensor in a closed local reference system, no matter what methods one applies, without exchanging signals with the external world? The answer is positive. In the following, we will explore this problem.

2. The Geodesic Deviation Equation

Let us choose a co-moving proper reference frame, an orthonormal tetrad, which consists of four mutually orthonormal base vectors, with the fourth vector coinciding with the unit tangent vector of the worldline (it is the geodesic in our present case). In this case, the tetrad is *parallelly transported* [6, 13, 14]. The four mutually orthonormal base vectors can be expressed as [6, 12]:

$$e_{(\alpha)} = \lambda^{\beta}_{(\alpha)} \frac{\partial}{\partial x^{\beta}} \tag{1}$$

where $\lambda^{\beta}_{(\alpha)}$ are the coefficients to be chosen. We note that, in this paper, *Einstein's summation convention* and *the light unit system* ($c \equiv 1$) are adopted; and furthermore, for Greek indices, the summation covers 0, 1, 2, 3; for Latin indices 1, 2, 3.

The orthonormality of a tetrad is given by the following condition

$$g^{\mu\nu} e_{(\alpha)\mu} e_{(\beta)\nu} = \eta_{(\alpha\beta)} \equiv \eta_{\alpha\beta} \tag{2}$$

where the index (α) denotes a specific vector (or tensor) and the index μ denotes the component with respect to coordinates x^{μ}, $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, which is the general metric tensor and reduces to the Minkowsky tensor $\eta_{\mu\nu} = diag(-1,1,1,1)$ if the spacetime becomes flat.

The general expression of the geodesic deviation can be written as follows [14, 15]:

$$a^{\mu} \equiv \frac{D^2 X^{\mu}}{Dt^2} = - R_{\lambda\alpha\sigma}{}^{\mu} T^{\lambda} X^{\alpha} T^{\sigma} \tag{3}$$

where D is *the covariant differential operator,* T^{λ} is *the tangent vector* to the geodesic, X^{μ} and a^{μ} denote *the diplacement vector* and the relative acceleration between two neighbouring geodesics, $R_{\lambda\alpha\sigma}{}^{\mu}$ is the Riemann tensor. Equation (3) gives the relativistic generalization of *the Newtonian tidal equation.*

With respect to the co-moving proper reference tetrad $e^{\mu}_{(\alpha)}$, the geodesic deviation equation can be expressed as [14,15]

$$\frac{d^2 X^{(\alpha)}}{dt^2} + \eta^{(\alpha\beta)} R_{(\varepsilon\rho\delta\beta)} T^{(\varepsilon)} X^{(\rho)} T^{(\delta)} = 0 \tag{4}$$

where

$$R_{(\varepsilon\rho\delta\beta)} = R_{\lambda\nu\sigma\mu} e^{\lambda}_{(\varepsilon)} e^{\nu}_{(\rho)} e^{\sigma}_{(\delta)} e^{\mu}_{(\beta)} \tag{5}$$

and

$$T^{(\lambda)} = \frac{dx^{(\lambda)}}{dt} \tag{6}$$

is the particle's 4-velocity observed in the chosen tetrad.

Equation (4) is similar to equation (3). The advantage of equation (4) is in that this equation is measurable in practice, at least partly. To confirm this conclusion, let us further investigate this equation.

3. The Determination of Riemannian Components

Suppose we use a gradiometer to measure the relative acceleration of two neighbouring geodesics, and let the gradiometer be at rest relative to the satellite, then, in the co-moving local reference frame, equation (4) becomes:

$$\frac{d^2X_{(0)}}{dt^2} = 0 \tag{7}$$

$$\frac{d^2X_{(i)}}{dt^2} + R_{(0j0i)}X^{(j)} = 0 \tag{8}$$

On the right hand side of equation (8), the first term is interpreted as the relative acceleration of the two proof masses and can be measured by gradiometers [1,4,6,9]. Equation (8) has the same form as the *tidal equation* expressed in the frame of Newtonian mechanics [8,9]

$$f_i = \frac{d^2\xi_i}{dt^2} = \frac{\partial^2 V}{\partial x^i \partial x^j}\,\xi^j$$

where f_i is the *tidal force*, $d^2\xi_i/dt^2$ is the relative acceleration (in the sense of Newtonian mechanics) of the two proof masses, and ξ^j is the distance between the two proof masses. This equation has been applied extensively in *satellite gradiometry* [5,10]. It can be shown that the equation (8) reduces to the above classical tidal equation under the Newtonian limit.

By appropriate orientations (puting the proof masses in the directions of $e_{(i)}$ respectively), we can make $R_{(0i0j)} = 0 (i \neq j)$ [6]. In this case, equation (8) becomes

$$\frac{d^2X_{(i)}}{dt^2} + R_{(0i0i)}X^{(i)} = 0 \quad \textit{(no sum over i!)} \tag{9}$$

where the relation

$$X^{(i)} = \eta^{(ij)}X_{(j)} = X_{(i)}$$

has been introduced. Hence, from (9) we can find the Riemannian components $R_{(0i0i)}$:

$$R_{(0i0i)} = -\left(\frac{d^2 X_{(i)}}{dt^2}\right)/X^{(i)} \quad \textit{(no sum over i!)} \tag{10}$$

and the remaining components $R_{(0i0j)} = 0 (i \neq j)$.

To find the Riemann tensor $R_{\lambda\nu\sigma\mu}$, we should apply equation (4), from which, noticing the orthonormality of the tetrad $e^{\lambda}_{(\alpha)}$:

$$e^{\lambda}_{(\alpha)} e^{(\alpha)}_{\beta} = \delta^{\lambda}_{\beta} \tag{11}$$

by multiplying both sides of the equation (5) with $e^{(\xi)}_{\xi} e^{(\rho)}_{\eta} e^{(\delta)}_{\theta} e^{(\beta)}_{\tau}$, we get:

$$R_{\mu\nu\alpha\beta} = R_{(\xi\eta\rho\sigma)}\, e^{(\xi)}_{\mu} e^{(\eta)}_{\nu} e^{(\rho)}_{\alpha} e^{(\sigma)}_{\beta} \tag{12}$$

We are most interested in R_{0i0j}, i.e.,

$$R_{0i0j} = R_{(\xi\eta\rho\sigma)}\, e^{(\xi)}_{0} e^{(\eta)}_{i} e^{(\rho)}_{0} e^{(\sigma)}_{j} \tag{13}$$

Suppose we can choose such a tetrad $e^{(\alpha)}$, so that the μ components $e^{(\alpha)}_{\mu} = 0$, if $\alpha \neq \mu$. Then, we have

$$R_{0i0j} = R_{(0i0j)} e^{(0)}_{0} e^{(i)}_{i} e^{(0)}_{0} e^{(j)}_{j} \quad \textit{(no sum over i, j!)} \tag{14}$$

Since gradiometers have been appropriately oriented ($R_{(i0j0)} = 0$, $i \neq j$, [6]), we have

$$R_{0i0j} = 0, \textit{ if } i \neq j \tag{15}$$

$$R_{0i0i} = R_{(0i0i)} e^{(0)}_{0} e^{(i)}_{i} e^{(0)}_{0} e^{(i)}_{i} \quad \textit{(no sum over i!)} \tag{16}$$

From (10), we can see that $R_{(0i0j)}$ are the quantities measured by gradiometers which are fixed on a satellite. Then, from (15) and (16), we can find R_{0i0j}, which are independent of the coordinate system. In this way, the gravitational effects are separated from inertia, at least partly. We should keep in mind that, according to the equivalence principle, in a freely moving elevator (or satellite), one can not tell whetherone is in the state of free fall or the state of uniform motion or at rest, no matter what method one uses, provided one does not exchange signals with the world outside the elevator. However, by some kinds of devices (such as gradiometers), one finds that the devices can "feel" the action of the gravitation (even if in a very small region provided the device can be made as small as possible). This conclusion is very attractive and it means that in a strict sense Einstein's equivalence principle is correct only at one point. Extending to any finite region, even if very small, the equivalence principle holds no more.

Now, let us explore how to determine R_{0i0j} from equation (16) or (13) in practice. In a general curved spacetime (four dimensional manifold), it is not easy to determine the base vectors of the orthonormal tetrad. However, with some kind of approximation, it becomes easier.

Let us introduce the standard PPN *coordinate induced tetrad* (at rest with respect to the coordinates) [12]:

$$e_{(t)} = \eta_{+}\frac{\partial}{\partial t}, \quad e_{(i)} = \eta_{-}\frac{\partial}{\partial x^i} \tag{17}$$

with

$$\eta_{\pm} = 1 \pm \frac{GM}{r} \tag{18}$$

where G is the gravitational constant and M is the earth's mass. Suppose we have chosen the spherical polar coordinate grid (t, r, θ, λ) with its origin at the earth's center, where r is the distance between the origin and the field point, θ is the polar angle, and λ is the longitude. The tangent vectors to the coordinate lines of the coordinate grid (t, r, θ, λ) are respectively

$\partial/\partial t$, $\partial/\partial r$, $\partial/\partial\theta$, $\partial/\partial\lambda$

Although the above tetrad is orthogonal, it is not parallelly transported (in general case). In fact, it is at rest with respect to the global coordinates (t, r, θ, λ). At every point P passed by the satellite, there exists a coordinate induced tetrad $e^{\mu}_{(\alpha)}$. However, in order to correlate the Riemann tensor with the measured quantities, we need to find the parallelly transported tetrad $e'^{\mu}_{(\alpha)}$, which is a proper reference frame of the satellite. For this purpose, we need to know the velocity of the satellite. Fortunately, in this case, the velocity is known.

Let us use $\vec{v}$ to denote the ordinary 3-velocity of the satellite observed in the geocentric star-fixed coordinate system (GSS). Then, the comoving parallelly transported tetrad $e'^{\mu}_{(\alpha)}$ can be obtained by a Lorentz transformation $\Lambda^{(\alpha)}_{(\beta)}$, i.e.,

$$e^{(\alpha)} = \Lambda^{(\alpha)}_{(\beta)} e^{(\beta)} \tag{19}$$

where

$$\begin{aligned} \Lambda^{(0)}_{(0)} &= 1 + \frac{1}{2}\upsilon^2 \\ \Lambda^{(i)}_{(0)} &= \Lambda^{(0)}_{(i)} = -\upsilon^i \equiv -\upsilon_i, \\ \Lambda^{(j)}_{(i)} &= \delta^j_i + \frac{1}{2}\upsilon^j\upsilon_i \end{aligned} \tag{20}$$

With the above tetrad, equation (12) or (13) should be used. In this case, we cannot find $R_{\mu\nu\alpha\beta}$ or R_{0i0j}, because only some components of $R_{(\xi\eta\rho\sigma)}$ have been measured. Hence, we need to apply an approximate method, with which the Riemann tensor can be found, and as a result the earth's gravitational field can be determined.

4. The Determination of the Gravitational Field

In the spacetime considered in section 3, suppose we have chosen a global spherical polar coordinate grid (t, r, θ, λ). In this case, at every fixed spacetime point P, there exists a *coordinate induced tetrad* $e^{(\alpha)}_{\mu}$ [12], which is given by expression (17).

It should be pointed out here that in general case one cannot find exact solution for determining a gravitational field. One must use anapproximate method. In our present case, if we use *the Post-Newtonian Approximation* [16], we will find that only five potential quantities need to be determined, where four of them can be calculated by a normal model (a uniform sphere) and the fifth is connected to the measured Riemannian components $R_{(0i0j)}$.

With the Post-Newtonian Approximation, the metric tensor $g_{\mu\nu}$ can be expressed as [16]

$$g_{00} = -1 - 2\phi - 2\phi^2 - 2\psi \tag{21}$$

$$g_{0j} = \zeta_j \tag{22}$$

$$g_{ij} = \delta_{ij} - 2\delta_{ij}\phi \tag{23}$$

where ϕ, ψ, ζ (which are on the orders υ^2, υ^3, υ^4) are the *first Newtonian potential, second Newtonian potential,* and *vector potential,* respectively.

The Riemannian components can be expressed as follows [16]:

$$\begin{aligned} R_{0i0j} &= \phi_{ij} + 3\phi_i\phi_j + 2\phi\phi_{ij} - \delta_{ij}(\nabla\phi)^2 + \psi_{ij}, \\ R_{0ijk} &= \frac{1}{2}(\partial_i\partial_j\zeta_k - \partial_i\partial_k\zeta_j) - \partial_t(\delta_{ij}\phi_k - \delta_{ik}\phi_j), \\ R_{ijkl} &= \delta_{ik}\phi_{jl} - \delta_{il}\phi_{jk} - \delta_{jk}\phi_{il} + \delta_{jl}\phi_{ik} \end{aligned} \tag{24}$$

where $\partial_i \equiv \partial/\partial x^i$, $\partial_t \equiv \partial/\partial t$, $\phi_i \equiv \partial_i\phi$, $\phi_{ij} \equiv \partial_i\partial_j\phi$, etc.

To determine the potential ϕ, we first establish the connection between $R_{\mu\nu\alpha\beta}$ and $R_{(\xi\eta\rho\sigma)}$. The connection between $R_{\mu\nu\alpha\beta}$ and $R_{(\xi\eta\rho\sigma)}$ can be easily established through the Lorentz transformation $\Lambda^{\mu\nu\alpha\beta}_{(\xi\eta\rho\sigma)}$:

$$R_{(\xi\eta\rho\sigma)} = \Lambda^{\mu\nu\alpha\beta}_{(\xi\eta\rho\sigma)} R_{\mu\nu\alpha\beta} \tag{25}$$

where

$$\Lambda^{\mu\nu\alpha\beta}_{(\xi\eta\rho\sigma)} \equiv \Lambda^{\mu}_{(\xi)}\Lambda^{\nu}_{(\eta)}\Lambda^{\alpha}_{(\rho)}\Lambda^{\beta}_{(\sigma)} \tag{26}$$

and $\Lambda^{\mu}_{(\xi)}$ is given by equation (20). Then, from the above two equations we have:

$$R_{(0i0j)} = R_{\mu\nu\alpha\beta}\Lambda^{\mu}_{(0)}\Lambda^{\nu}_{(i)}\Lambda^{\alpha}_{(0)}\Lambda^{\beta}_{(j)} \tag{27}$$

From (20) and (27), accurate to υ^4, we find that the following connection holds:

$$\begin{aligned} R_{(0i0j)} = {} & R_{0i0j}(1+\upsilon^2) - \frac{1}{2}R_{0i0k}\upsilon^k\upsilon_j\frac{1}{2} - R_{0k0j}\upsilon^k\upsilon_i \\ & - R_{0ikj}\upsilon^k - R_{0jki}\upsilon^k + R_{kimj}\upsilon^k\upsilon^m \end{aligned} \tag{28}$$

If substituting (24) into (28), we find that equation (28) connects the measured quantities $R_{(0i0j)}$ with the potentials ϕ, ζ_i and ψ. Obviously, it is impossible to find the potentials by using the measured quantities $R_{(0i0j)}$ at only one point. However, by using a lot of sets of quantities $R_{(0i0j)}$ (observed in one or more satellite orbits) it may be possible to determine the potentials. To confirm this conclusion, let us first consider the Newtonian limit: accurate to υ^2. In this case, from (28) and the first equation of (24) we get

$$R_{0i0j} = R_{(0i0j)} = \phi_{ij} \tag{29}$$

Now since the components R_{0i0j} have been found, we can use the conventional method [10] to determine the first Newtonian potential ϕ. The basic idea is as follows:

Set

$$V = -\phi \tag{30}$$

in agreement with the common concept of the potential used in geodesy. Combining (29) and (30) we get

$$\partial_i \partial_j V = - R_{(0i0j)} \quad (31)$$

Suppose that in the region where the mass density ρ vanishes the potential V can be expressed as a *spherical harmonic expansion* [10]:

$$V = C + C_i x^i + \frac{GM}{R} \sum_{n=0}^{N} \sum_{m=0}^{n} \sum_{\alpha=0}^{1} C_{nm\alpha} \left(\frac{R}{r}\right)^{n+1} Y_{nm\alpha}(P) \quad (32)$$

where C and C_i are constants to be determined (if only considering the earth's potential, C and C_i are equal to zero), $C_{nm\alpha}$ are unknown potential coefficients to be determined, N is a large enough possitive integer (depending on the accuracy required), and

$$Y_{nm0}(P) = P_{nm}(\cos\theta)\cos(m\lambda),$$
$$Y_{nm1}(P) = P_{nm}(\cos\theta)\sin(m\lambda) \quad (33)$$

are fully normalized spherical harmonics of degree n and order m.

Substituting equation (32) into (31) we get

$$\frac{GM}{R} \sum_{n=0}^{N} \sum_{m=0}^{n} \sum_{\alpha=0}^{1} C_{nm\alpha} \partial_i \partial_j \left[\left(\frac{R}{r}\right)^{n+1} Y_{nm\alpha}(P)\right] = - R_{(0i0j)} \quad (34)$$

Note that point P is on the trajectory (orbit) of the satellite. With equation (34), in principle, the coefficients $C_{nm\alpha}$ can be determined, provided sufficiently many sets of the Riemannian components R_{0i0j} are observed. (For practical reasons, greater and well determined coefficients, corresponding to a reference gravity field, should be taken out first.)

Suppose we have determined all the coefficients $C_{nm\alpha}$ from equation (34), then, from equation (32) we know that the potential V has been determined up to four constants C and C_i. These four constants cannot be determined by the Riemannian components R_{0i0j}. However, in the case of determining the gravity field of the earth, we require that the potential V approaches zero with P tending to *infinity;* then, $C = C_i = 0$. In this sense, the potential V is completely determined [10].

Now we aim at the accuracy of υ^4. From (28) and (24) we get:

$$V_{ij} \equiv - \phi_{ij} = - R_{(0i0j)} + Q_{ij} \quad (35)$$

where

$$Q_{ij} = \{[3U_i U_j + 2U U_{ij} - \delta_{ij} (\nabla U)^2 + \psi_{ij}] - [2U_{ij}\upsilon^2 - \frac{3}{2} U_{ik}\upsilon^k \upsilon_j - \frac{3}{2} U_{kj}\upsilon^k \upsilon_i + \delta_{ij} U_{km}\upsilon^k \upsilon^m] + [\partial_i \partial_j \zeta_k - \frac{1}{2} \partial_i \partial_k \zeta_j - \frac{1}{2} \partial_j \partial_k \zeta_i]\} \quad (36)$$

U (the normal spherical potential) as well as ζ_i and ψ are calculated by a normal spherical model [11]. In this sense, V_{ij} have been determined as the measured quantities.

Substituting equation (32) into (35), we get a relativistic model of satellite gradiometry (accurate to the order of υ^4)

$$\frac{GM}{R} \sum_{n=0}^{N} \sum_{m=0}^{n} \sum_{\alpha=0}^{1} C_{nm\alpha} \partial_i \partial_j \left[\left(\frac{R}{r}\right)^{n+1} Y_{nm\alpha}\right] = - R_{(0i0j)} + Q_{ij} \quad (37)$$

which is a generalization of the Newtonian model (34).

5. Proofs of Two Statements

Now, we put forward the following two statements:

1. *In the case of using gradiometers on a satellite, with some kind of approximation, the Riemann tensor $R_{\mu\nu\alpha\beta}$ can be found.*
2. *In the case of free motion, if the measured Riemannian components $R_{(0k0l)}$ are always equal to zero, then, accurate to υ^2, the whole Riemann tensor $R_{\mu\nu\alpha\beta}$ equals zero.*

Let us first prove the second statement. Suppose

$$R_{(0k0l)} = 0 \quad (38)$$

From (34) and (38), the potential coefficients $C_{nm\alpha}$ must be equal to zero:

$$C_{nm\alpha} = 0 \quad (39)$$

Then, from equation (32) we have

$$V = C + C_i x^i$$

This potential denotes a global uniform gravitational field. However, in reality, no global uniform gravitational field exists; hence we have

$$V = C$$

or without loss of generality, we choose $C = 0$, i.e.

$$\phi = - V = 0 \quad (40)$$

This equation means that there exists no mass sources in our spacetime. Consequently we have

$$\phi = 0, \vec{\zeta} = 0, \psi = 0 \quad (41)$$

In this case, we can choose a global coordinate system in which the metric tensor is the Minkowsky metric $\eta_{\mu\nu} = diag(-1,1,1,1)$. In this coordinate system, the Riemann tensor $R_{\mu\nu\alpha\beta}$ is equal to zero:

$$R_{\mu\nu\alpha\beta} = 0 \quad (42)$$

Since any tensor is invariant under coordinate transformations, the above equation holds in any coordinates, i.e., the Riemann tensor $R_{\mu\nu\alpha\beta}$ is always equal to zero. This completes the proof of the second statement.

To prove the first statement, we need only to prove that the metric tensor $g_{\mu\nu}$ can be determined with the aid of the Post-Newtonian Approximation. In fact, we have shown (see section 4) that the first Newtonian potential ϕ can be determined provided that as many sets of the Riemannian components $R_{(0k0l)}$ as possible are observed. As a result. equations (21), (22) and (23) tell us that the metric tensor $g_{\mu\nu}$ can be determined. Once $g_{\mu\nu}$ is determined, $R_{\mu\nu\alpha\beta}$ is also determined. This completes the proof.

6. Discussion

In principle, the relativistic gravity field can be determined strictly by satellite gradiometry. The method is very general. First, we determine ϕ, ψ and ζ_i by measuring as many sets of the components $R_{(0k0l)}$ as possible. Now ϕ, ψ and ζ_i are connected with $R_{\mu\nu\alpha\beta}$ by equation (24) and $R_{\mu\nu\alpha\beta}$ is connected with $R_{(0k0l)}$ through Lorentz transformations; hence ϕ, ψ and ζ_i (using spherical harmonic expansion forms) are connected with $R_{(0k0l)}$. In this way, ϕ, ψ and ζ_i can be determined and consequently $g_{\mu\nu}$ can be found by equations (21), (22) and (23). Once the metric tensor $g_{\mu\nu}$ is determined, as a result the (relativistic) gravitational field expressed in the geocentric star-fixed system (GSS) is determined. This field, in essence, is just the gravity field expressed in the geocentric earth-fixed system (GES). To transform the expression from GSS to GES is straightforward.

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